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c-Extensions of P- and T-Geometries

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We consider affine extensions of geometries of Petersen and of tilde type, i.e., of flag-transitive geometries Γ belonging to a diagram

$$(c.X)\qquad \overset{1}{\circ}\overset{c}{\text{———}}\overset{2}{\circ}\cdots\overset{n-2}{\circ}\text{———}\overset{n-1}{\circ}\overset{X}{\text{———}}\overset{n}{\circ},$$

where either $\circ\overset{X}{\text{——}}\circ=\circ\overset{P*}{\text{——}}\circ$, i.e., the geometry of edges and vertices of the Petersen graph, or $\circ\overset{X}{\text{——}}\circ=\circ\overset{\sim}{=\!=}\circ$, i.e., the 3-fold cover of the generalized $Sp_4(2)$-quadrangle. We call such geometries $(c.P)$- resp. $(c.T)$-geometries because the residue of an element of type 1 is a P- resp. T-geometry. The class of flag-transitive P- and T-geometries has been classified by different authors and an overview of the results is given in [7]. In particular, there exist only two flag-transitive P-geometries of rank 3 with automorphism groups $\mathrm{Aut}(M_{22})$ and $3\mathrm{Aut}(M_{22})$ and three flag-transitive T-geometries of rank 3 with automorphism groups M_{24}, He, and $3^7Sp_6(2)$. In this paper we will assume:

$(*)$ If F is a flag of Γ of type $\{1, ..., n-3\}$ then the residue $\mathrm{res}(F)$ of F in Γ is the P-geometry for M_{22} or the T-geometry for M_{24}.

Since there are no P-geometries of rank $\geqslant 5$ and no T-geometries of rank $\geqslant 5$ satisfying $(*)$, this forces $n\leqslant 6$. On the other hand, it is known that for $n=3$ the universal covers of the rank 3 $(c.P)$- and $(c.T)$-geometry are both infinite; so a general classification for $n=3$ will not be possible. The case of flag-transitive $(c.P)$-geometries Γ of rank 4 and satisfying $(*)$ has been considered in [4]. There it is shown that $\mathrm{Aut}(\Gamma)$ is isomorphic to a factorgroup of one of the groups $2^{11}\colon \mathrm{Aut}(M_{22})$, M_{24} or $2\cdot U_6(2)\colon 2$ (where by $H:K$ and $H\cdot K$ we denote, as usual, a split resp. non-split extension of a group K by a group H). In the first case, the normal 2-subgroup is the universal representation group (see below) of the P-geometry for M_{22}. It is isomorphic to a submodule of index two (the even half) of the

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Golay co-code. Examples for the other two geometries are described in [4].

DEFINITION 1 (see [6]). Let Γ be any geometry which contains some sets of objects $\mathscr{P}$ and $\mathscr{L}$ ("points" and "lines") such that any line $l \in \mathscr{L}$ is incident to exactly three points of $\mathscr{P}$. Let $U(\Gamma)$ be the abstract group defined by

$$U(\Gamma) = \langle u_x \mid x \in \mathscr{P},\ u_x^2 = 1 = u_x u_y u_z \text{ whenever } x, y, z \text{ are the three points incident to a line } l \in \mathscr{L} \rangle.$$

Then $U(\Gamma)$ is called the *universal representation group* of the point-line system $(\mathscr{P}, \mathscr{L})$.

If Γ is a P- or T-geometry with diagram $\overset{1}{\circ}\overset{c}{\text{———}}\overset{2}{\circ}\cdots\overset{n-2}{\circ}\text{———}\overset{n-1}{\circ}\overset{X}{\text{———}}\overset{n}{\circ}$ we define the group $U(\Gamma)$ to be the universal representation group of the point-line system $(\mathscr{P}, \mathscr{L})$ where $\mathscr{P}$ and $\mathscr{L}$ are the objects of Γ of type 1 and 2.

Of course, the group $U(\Gamma)$ might become trivial, as it does, e.g., in case of the rank 4 P-geometry for M_{23}. But if it is nontrivial, then $\mathrm{Aut}(\Gamma)$ acts on $U(\Gamma)$ in a natural way and the semidirect product of $U(\Gamma)$ with $\mathrm{Aut}(\Gamma)$ acts flag-transitively on a c-extension of Γ.

The universal representation groups of the P- and T-geometries occurring as residues in this paper, are all known (see [6]). This was in a certain sense the reason for us to restrict our considerations in the present paper to geometries satisfying $(*)$. In the case of $(c.P)$-geometries, we will use an induction argument where the induction is anchored in [4] but we do not have any equivalent result for the case of $3 \cdot M_{22}$. In fact, the proof in [4] consists in finding a presentation in terms of generators and relations for any flag-transitive $G \leqslant \mathrm{Aut}(\Gamma)$ and then to identify G via computer calculations. However, the same approach must necessarily fail in the case of $3 \cdot M_{22}$ because already the known examples shown that G can become very large. Furthermore, at present it is not even known whether the universal representation groups of $3 \cdot M_{22}$, $3^{23}Co_2$, $3^{3471}BM$, $3^7Sp_6(2)$, and He are finite or infinite.

The universal representation group of the P-geometry for Co_2 is an elementary abelian 2-group of order 2^{23}, and it is isomorphic to the submodule of index 2 stabilized by Co_2 in the Leech lattice mod 2. This module contains a one-dimensional subspace invariant under the action of Co_2 and the quotient over this subspace is also a representation group [6, (5.7)]. The representation groups of the T-geometries for M_{24} and Co_1 are elementary abelian of order 2^{11} resp. 2^{24} and isomorphic to the 11-dimensional irreducible submodule of the Golay co-code resp. to the Leech lattice mod 2 [6, (3.2), (5.8)]. Also by [6] the universal representation group of the P-geometry for the Baby Monster sporadic simple group BM is a non-split extension $2 \cdot BM$ and the universal representation group of the T-geometry for the Monster M is isomorphic to M. By an unpublished result of A. Ivanov and S. Shpectorov there exist examples of c-extensions with automorphism groups $2 \cdot (BM \times BM): 2$ and $(M \times M): 2$, which are related to the representation group in these cases as well.

The aim of the present paper is to show that under the assumption of $(*)$ all flag-transitive $(c.T)$-geometries arise in the context of the representation group and that there are only three more examples of $(c.P)$-geometries, which we will describe in Section 1. More precisely we prove

THEOREM 1. *Let $G \leqslant \mathrm{Aut}(\Gamma)$ be a flag-transitive automorphism group of a $(c.P)$-geometry Γ of rank $n \geqslant 5$. Assume that the residue of each flag of type $\{1, ..., n-3\}$ is the P-geometry for M_{22}.*

(a) *If $n=5$ and the residue of an element of type 1 is the P-geometry for M_{23} then $G \cong M_{24}$.*

(b) *If $n=5$ and the residue of an element of type 1 is the P-geometry for Co_2 then either G is isomorphic to one of the groups $2^{22}: Co_2$, $2^{23}: Co_2$, and $O_2(G)$ considered as module for $G/O_2(G)$ is isomorphic to a section of the Leech lattice mod 2; or $G \cong Co_1$.*

(c) *If $n=6$ then either G is isomorphic to one of the groups $BM \times BM$, $BM \wr \mathbf{Z}_2$, $2 \cdot (BM \times BM)$, $2 \cdot (BM \times BM): 2$, or $G \cong M$.*

THEOREM 2. *Let $G = \mathrm{Aut}(\Gamma)$ be the automorphism group of a $(c.T)$-geometry Γ of rank $n \geqslant 4$. Assume that the residue of each flag of type $\{1, ..., n-3\}$ is the tilde geometry for M_{24}.*

(a) *If $n=4$ then G is the semidirect product of M_{24} with the* (11-*dimensional*) *Golay code module.*

(b) *If $n=5$ then G is the semidirect product of Co_1 with the Leech lattice mod 2.*

(c) *If $n=6$ then $G \cong M \times M$, or $M \wr \mathbf{Z}_2$.*

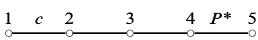
We prove Theorems 1 and 2 in a series of lemmas in Section 4. In Section 2 we derive some general results about subgeometries of Γ, which apart from being useful for the determination of $\mathrm{Aut}(\Gamma)$ might also be interesting on their own. Section 3 contains some further preliminary results.

For the rest of the paper we fix the following notation: Γ is a $(c.P)$- or a $(c.T)$-geometry, $\Gamma^{(i)}$ is the set of objects of Γ of type i, $\{\alpha_1, ..., \alpha_n\}$ is a maximal flag in Γ with $\alpha_i \in \Gamma^{(i)}$, $G \leqslant \mathrm{Aut}(\Gamma)$ is a flag-transitive automorphism group of Γ, $G_i = G_{\alpha_i}$ is the stabilizer of α_i in G, K_i is the kernel of the action of G_i on the residue $\operatorname{res}(\alpha_i)$ of α_i in Γ, $P_i = \bigcap_{j \neq i} G_j$ are the so-called "minimal parabolics" and $B = P_1 \cap \cdots \cap P_n$ is the "Borel subgroup". Elements of type 1 will sometimes be called "points", elements of type 2 "lines", and elements of type 3 "planes." Further, for $x \in \Gamma^{(i)}$, by $\operatorname{res}(x)^-$ we denote the geometry induced on the objects of types 1, ..., $i-1$ in $\operatorname{res}(x)$ and by $\operatorname{res}(x)^+$ the geometry induced on the objects of types $i+1$, ..., n.

By 2^n we usually denote an elementary abelian 2-group of order 2^n, by $[2^n]$ any 2-group of order 2^n, and by $2^{n_1+n_2+\cdots+n_k}H$ we mean the extension of a 2-group P by a group H such that P has an H-invariant normal series $1 = P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_k = P$ with $P_i/P_{i-1} \cong 2^{n_i}$. The rest of the notation is standard.

1. EXAMPLES OF $(c.P)$-GEOMETRIES

In this section we describe the examples of $(c.P)$-geometries occurring in our paper that do not correspond to a representation group of the residual P-geometry. As far as we know these geometries have not already appeared in literature. In a forthcoming paper we will describe them in a wider context. Here we just sketch them, i.e., we do not provide detailed proofs. But with some knowledge about Mathieu groups and their Steiner systems, Conway groups and the Leech lattice, and the Monster and Baby Monster sporadic simple groups the interested reader should be able to verify by himself that the described geometries are flag-transitive and of $(c.P)$-type. Anyway the description given below should be sufficient for the purpose of this paper.

1.1. *A Geometry for M_{24} with Diagram* 

Let $\mathscr{S} = \mathscr{S}(24, 8, 5)$ be the Steiner system for $G = M_{24}$ with underlying set Ω (for some properties of $\mathscr{S}$ see, e.g., [2, Chaps. 6, 7]). Let $\mathscr{O}$ denote the set of octads and $\mathscr{T}$ the set of trios. We set

— $\Gamma^{(1)} = \Omega$,

— $\Gamma^{(2)} = \{\{a_1, a_2\} \mid a_i \in \Omega,\, a_1 \neq a_2\}$,

— $\Gamma^{(3)} = \{\{a_1, a_2, a_3, a_4\} \mid a_i \in \Omega,\, a_i \neq a_j \text{ for } i \neq j\}$,

— $\Gamma^{(4)} = \{(O_1, \{O_2, O_3\}) \mid \{O_1, O_2, O_3\} \in \mathscr{T}\}$,

— $\Gamma^{(5)} = \mathscr{O}$.

Incidences between objects of types 1, 2, 3 are defined by inclusion. An element $a \in \Gamma^{(1)}$ is incident to $(O_1, \{O_2, O_3\}) \in \Gamma^{(4)}$ if $a \in O_1$, and to $O \in \Gamma^{(4)}$ if $a \notin O$. Elements $x \in \Gamma^{(2)} \cup \Gamma^{(3)}$ and $y \in \Gamma^{(4)} \cup \Gamma^{(5)}$ are incident if all elements of x are incident to y. The elements of type 5 in res(x) for $x=(O_1, \{O_2, O_3\}) \in \Gamma^{(4)}$ are O_2 and O_3.

1.2. *A geometry for* Co_1 *with diagram* $\overset{1}{\circ}\overset{c}{\text{——}}\overset{2}{\circ}\text{——}\overset{3}{\circ}\text{——}\overset{4}{\circ}\overset{P^*}{\text{——}}\overset{5}{\circ}$

In order to define the geometry for Co_1 we first briefly recall the construction of the P-geometry for Co_2 as it is given in [9].

Let Λ denote the Leech lattice with inner product $(\ ,\ )$ and let $\Lambda_2=\{v \in \Lambda, (v, v)=4\}$. Consider the group $H=Co_2$ as the stabilizer in $Aut(\Lambda)$ of a fixed vector $v_0 \in \Lambda_2$. Let $\Sigma=\{\{v, -v\} \mid v \in \Lambda_2, (v_0, v)=0\}$ and define a graph on Σ so that σ and $\tau \in \Sigma$ are adjacent if $H_\sigma \cap H_\tau \cong 2^9.2^5\Sigma_5$. A clique $\mathscr{C}$ in this graph is called closed if for each $\{v, -v\}, \{w, -w\} \in \mathscr{C}$ there exists $\{u, -u\} \in \mathscr{C}$ such that $v_0+v+w+u \in 2\Lambda$. It is shown in [9] that each closed clique is of size 1, 3, 7, or 15, and that, if we take as objects of type i the closed cliques of size 2^i-1 and define incidence by inclusion, then we get the P-geometry for Co_2.

Now we can define the $(c.P)$-geometry for Co_1: Let

— $\Gamma^{(1)}=\{\{v, -v\} \mid v \in \Lambda_2\}$.

For $x=\{v, -v\} \in \Gamma^{(1)}$ let $\Sigma_x=\{\{w, -w\} \in \Gamma^{(1)} \mid (v, w)=0\}$ and consider Σ_x as the graph just described. Let

— $\Gamma^{(i)}=\{\{x\} \cup \mathscr{C} \mid x \in \Gamma^{(1)}, \mathscr{C} \subseteq \Sigma_x, |\mathscr{C}|=2^{i-1}-1,$ $\mathscr{C}$ is a closed clique in the graph on $\Sigma_x\}$, for $2 \leqslant i \leqslant 5$,

and define incidence by inclusion. Then it is straightforward to show that $\Gamma=\bigcup_{i=1}^5 \Gamma^{(i)}$ is a geometry with the desired diagram. Since Co_1 is transitive on $\{\{v, -v\} \mid v \in \Lambda_2\}$, it is clear that $\mathrm{Aut}(\Gamma) \cong Co_1$.

1.3. *A geometry for the Monster with diagram* $\overset{1}{\circ}\overset{c}{\text{——}}\overset{2}{\circ}\text{——}\overset{3}{\circ}\text{——}\overset{4}{\circ}\text{——}\overset{5}{\circ}\overset{P^*}{\text{——}}\overset{6}{\circ}$

The idea of the construction is quite similar to the previous example. Recall that $2 \cdot BM=C_M(z)$ for an involution $z \in M$ of type $2A$ in the notation of [3] and that the objects of the P-geometry for BM can be identified with the conjugacy classes of elementary abelian 2-subgroups of BM of orders 2, 4, 8, 16 containing only involutions of type $2B$ resp. with one of the two such classes of order 32 (see, e.g., [6]). Let us denote these classes by $\mathscr{C}_1, \ldots, \mathscr{C}_5$. Then we can define Γ in the following way:

— $\Gamma^{(1)}=2A$,

— $\Gamma^{(i)}=\{zU \mid z \in 2A,\ U \leqslant C_M(z),\ U$ is elementary abelian of order 2^{i-1}, and $\langle U, z\rangle/\langle z\rangle \in \mathscr{C}_{i-1}\}$, for $2 \leqslant i \leqslant 6$

(where the $\mathscr{C}_i$ are to be understood as the corresponding conjugacy classes in $C_M(z)/\langle z\rangle \cong BM$). Incidence is again defined by inclusion.

If $zU \in \Gamma^{(i)}$ for some $i > 1$ and $u \in U$, then u is of type $2B$ in M, i.e. $C_M(u) \cong 2^{1+24}Co_1$, and $\langle z, U\rangle \leqslant O_2(C_M(u))$. Since $O_2(C_M(u))$ is extraspecial and $N_M(U) \cap C_M(z)$ is transitive on U, $N_M(\langle z, U\rangle)$ is transitive on zU. In particular, $|N_M(\langle z, U\rangle)\colon N_M(\langle z, U\rangle) \cap C_M(u)| = 2^{i-1}$ and $N_M(\langle z, U\rangle)$ acts as $2^{i-1}L_{i-1}(2)$ on $\langle z, U\rangle$. On the other hand, the elements in $\operatorname{res}^+(zU)$ are of the shape zW with $zU \subseteq zW$ which gives an isomorphism onto a P-geometry of rank $6-i$. From this it is straightforward to see that Γ is a geometry of $(c.P)$-type.

2. SUBGEOMETRIES

In this section Γ always denotes a $(c.P)$-geometry of rank $n \geqslant 4$. The reader might take it as an exercise to generalize the results to any flag-transitive geometry with diagram $\overset{1}{\circ}\overset{c}{\text{———}}\overset{2}{\circ}\cdots\overset{n-2}{\circ}\text{———}\overset{n-1}{\circ}\overset{X}{\text{———}}\overset{n}{\circ}$, where $\overset{j}{\circ}\overset{X}{\text{———}}\overset{j}{\circ}$ may be an arbitrary rank 2 geometry in which every object of type j is incident to exactly three objects of type i. Further one should assume that the stabilizer of an element $x \in \Gamma$ of type $i \geqslant 2$ induces the full affine linear group $2^{i-1}L_{i-1}(2)$ on $\operatorname{res}(x)^-$ (or something similar). We will use in particular the fact that G_x acts 3-fold transitively on the set of points and 2-fold transitively on the set of partitions of $\operatorname{res}(x)^-$ into classes of parallel lines. (This is automatically true if $X = P^*$.)

2.1. *Shrinking*

In this subsection we define a new incidence structure $\mathcal{S}$ related to Γ. In general, the structure $\mathcal{S}$ might not be connected (in fact, for most of the $(c.P)$-geometries it is disconnected, and these are the cases we are really interested in), but it is not difficult to see that its connected components are geometries with diagram of type $(c.P)$ (resp. $(c.X)$) but of rank $n-1$. (Therefore, we call this construction "shrinking".)

The structure $\mathcal{S}$ has the following sets of objects:

— $\mathcal{S}^{(1)} = \Gamma^{(2)}$,

— $\mathcal{S}^{(i)} = \{(x, P_x) \mid x \in \Gamma^{(i+1)},\ P_x$ is a partition of $\operatorname{res}(x)^-$ into parallel lines$\}$,

for $2 \leqslant i \leqslant n-1$. The elements $(x, P_x) \in \mathcal{S}^{(i)}$, $(y, P_y) \in \mathcal{S}^{(j)}$ with $i, j > 1$, are incident if x and y are incident in Γ and P_x is compatible with P_y; $x \in \mathcal{S}^{(1)}$ and $(y, P_y) \in \mathcal{S}^{(i)}$, $i > 1$, are incident if x is incident to y in Γ and is part of the partition P_y (recall that x is a line).

In the following we derive some properties of $\mathcal{S}$. The first lemma is due to A. Pasini.

LEMMA 1. *Let $\mathcal{S}$ be disconnected. Then, for every $x \in \Gamma^{(i)}$, $i \geqslant 3$, and two different partitions P_1, P_2 of* $\operatorname{res}(x)^-$ *into parallel lines, the objects (x, P_1) and (x, P_2) belong to different connected components of $\mathcal{S}$.*

Proof. The relation "giving rise to objects in the same connected component" is clearly an equivalence relation on the set of partitions of $\operatorname{res}(x)^-$ into parallel lines and it is preserved by the action of G_x on that set. But as mentioned above that action is 2-transitive and therefore primitive. So either we have the statement or all partitions give rise to objects in the same connected component of $\mathcal{S}$. However, in the latter case the connectedness of Γ forces $\mathcal{S}$ to be connected. ∎

We can also describe the shrinking in terms of the corresponding parabolic subgroups: If $P_1, ..., P_n$ are the minimal parabolics corresponding to a fixed maximal flag $\{\alpha_1, ..., \alpha_n\}$ of Γ and $R_1, ..., R_{n-1}$ the minimal parabolics belonging to a certain maximal flag in a suitable connected component Σ of $\mathcal{S}$, then $R_{\mathrm{i}} = \langle P_1, P_{\mathrm{i}+1} \rangle$ for $i \geqslant 2$ and $R_1 = N_{\langle P_1, P_2 \rangle}(P_1)$. If G_Σ denotes the stabilizer of Σ in G, then clearly $G_\Sigma = \langle R_1, ..., R_{n-1} \rangle$ and $\mathcal{S}$ is connected iff $G = G_\Sigma$. For the Borel-subgroup B_Σ we have $B_\Sigma = R_1 \cap \cdots \cap R_{n-1} = P_1$, hence $|B_\Sigma : B| = 2$. The maximal flag of Σ stabilized by B_Σ is of the shape

$$\{\alpha_2, (\alpha_3, P_3), ..., (\alpha_n, P_n)\},$$

where $P_3, ..., P_n$ are the partitions of $\operatorname{res}^-(\alpha_3), ..., \operatorname{res}^-(\alpha_{\mathrm{n}})$ into lines parallel to α_2.

LEMMA 2. *Let G_Σ be the stabilizer of a connected component Σ of $\mathcal{S}$, $N \trianglelefteq G_\Sigma$ the kernel of the action of G_Σ on Σ and $H = G_\Sigma/N$. Suppose*

(a) *the kernel of the action of H_p on the residue of p in Σ is trivial for $p \in \Sigma$ of type* 1,

(*b*) $G_2/K_2 \cong \mathbf{Z}_2 \times G_{12}/K_2$.

Let $Q_2 \trianglelefteq G_2$ such that $|Q_2 \colon K_2| = 2$ and $G_2 = Q_2 G_{12}$. Then $N = Q_2$.

Proof. We identify $p \in \Sigma$ with $\alpha_2 \in \Gamma$ and H_p with G_2N/N. Now Q_2 is trivial on the elements of types 3, ..., n of $\operatorname{res}(\alpha_2)$ in Γ, hence it is trivial on the residue of p in Σ. By assumption (a) $Q_2 \leqslant N$.

Let F_Σ be a maximal flag in Σ. Then F_Σ is stabilized by N. By construction, F_Σ comes from a unique flag F of cotype 1 in Γ. So N stabilizes F. Since F is contained in exactly two maximal flags, we have $|N : N \cap B| \leqslant 2$. On the other hand, our construction implies that $N \cap B \leqslant K_2$. So the assertion follows from $Q_2 \leqslant N$ and $|Q_2 : K_2| = 2$. ∎

COROLLARY 1. *Suppose the assumptions of Lemma 2 hold. Then $\mathscr{S}$ is disconnected.*

Proof. By Lemma 2 $Q_2 = N \trianglelefteq G_\Sigma$. Since $Q_2 \leqslant G_2$, Q_2 must stabilize each element of $\{\alpha_2^{G_\Sigma}\}$. Hence $G_\Sigma \neq G$. ∎

If $n \geqslant 5$, we can repeat the shrinking construction applying it to $\mathscr{S}$. Then we get a new incidence structure, say $\mathscr{T}$, whose connected components are contained in the connected components of $\mathscr{S}$ and constitute $(c.P)$-geometries of rank $n-2$. For the set $\mathscr{T}^{(1)}$ of objects of $\mathscr{T}$ of type 1 we have by construction

$$\mathscr{T}^{(1)} = \mathscr{S}^{(2)} = \{(x, P_x) \mid x \in \Gamma^{(3)}, P_x \text{ is a partition of } \operatorname{res}(x)^- \text{ into parallel lines}\}.$$

Now, for each $x \in \Gamma^{(3)}$, there are exactly three different partitions P_1, P_2, P_3 of $\operatorname{res}(x)^-$ into two parallel lines, and by Lemma 1 either $\mathscr{S}$ is connected or each pair (x, P_i) belongs to a different connected component of $\mathscr{S}$, hence also to a different connected component of $\mathscr{T}$. Define a relation $\equiv$ on the set of connected components of $\mathscr{T}$ by $\Theta_1 :\equiv \Theta_2$ if there exist $x \in \Gamma^{(3)}$ and partitions P_1, P_2 of $\operatorname{res}(x)^-$ into parallel lines such that (x, P_i) belongs to Θ_i for $i = 1, 2$. Then the following holds.

LEMMA 3. *Suppose $\mathscr{S}$ is disconnected and define the relation $\equiv$ as just described. If $\Theta_1 \equiv \Theta_2$ then the elements of $\Theta_1^{(1)}$ and $\Theta_2^{(1)}$ arise from the same set of elements from $\Gamma^{(3)}$. Moreover, $\equiv$ is an equivalence relation on the set of connected components of $\mathscr{T}$, each equivalence class is of size three and, if $\{\Theta_1, \Theta_2, \Theta_3\}$ is such an equivalence class, then the Θ_i are contained in different connected components of $\mathscr{S}$.*

Proof. Obviously $\equiv$ is reflexive and symmetric. Moreover, if the first statement of the lemma is shown, then it is clear that $\equiv$ is also transitive, hence an equivalence relation, and Lemma 1 together with the fact that, for any $x \in \Gamma^{(3)}$, there are exactly three partitions of $\operatorname{res}(x)^-$ into parallel lines implies the rest of the assertions.

Let $\Theta_1 \equiv \Theta_2$ and $x \in \Gamma^{(3)}$ such that $(x, P_i) \in \Theta_i$ for certain partitions P_1, P_2 of $\operatorname{res}(x)^-$. By $\operatorname{res}(x, P_i)^{(2)}$ denote the set of elements of type 2 in the residue of (x, P_i) in Θ_i. It follows from the construction of $\mathscr{T}$ that

$$\operatorname{res}(x, P_i)^{(2)} = \{(l, P_{S,l}) \mid l \in \mathscr{S}^{(3)}, l \text{ is incident to } (x, P_i) \text{ in } \mathscr{S}, \text{ and } (x, P_i) \text{ is part of the partition } P_{S,l} \text{ of the residue of } (l, P_{S,l}) \text{ into parallel lines of } \mathscr{S}\}.$$

As $l \in \mathscr{S}^{(3)}$ and l is incident to (x, P_i), by construction of $\mathscr{S}$ there is $l_0 \in \Gamma^{(4)} \cap \operatorname{res}(x)$ and a partition P_0 of $\operatorname{res}(l_0)^-$ into parallel lines of Γ such that $l = (l_0, P_0)$ and P_0 is compatible with the partition P_i. In terms of Γ

then x is a plane of l_0 and $P_{S,l}$ induces a partition of $\mathrm{res}(l_0)^-$ into two parallel planes. If $(x', P_i') \in \Theta_i^{(1)}$ is the unique element incident to $(l, P_{S,l})$ in Θ_i and different from (x, P_i), then x' is also a plane of $\mathrm{res}(l_0)^-$ and (x', P_i') must be part of the partition $P_{S,l}$. So x' must be the plane in $\mathrm{res}(l_0)^-$ parallel to x. This means that there is a bijection between the set of points collinear to (x, P_i) in Θ_i and the planes parallel to x in Γ. Since Θ_i is connected, this implies the assertion. ∎

Let $\bar{\mathscr{T}}$ be the incidence structure with sets of objects

— $\bar{\mathscr{T}}^{(1)} = \Gamma^{(3)}$,

— $\bar{\mathscr{T}}^{(i)} = \{(x, P_x) \mid x \in \Gamma^{(i+2)},\ P_x$ is a partition of $\mathrm{res}(x)^-$ into parallel planes$\}$,

for $2 \leqslant i \leqslant n-2$, and incidence defined similarly as for $\mathscr{S}$, i.e., $x \in \bar{\mathscr{T}}^{(1)}$ and $(y, P_y) \in \bar{\mathscr{T}}^{(i)}$, $i > 1$, are incident if x and y are incident in Γ and x is part of the partition P_y, and $(x, T_x) \in \bar{\mathscr{T}}^{(i)}$, $(y, P_y) \in \bar{\mathscr{T}}^{(j)}$ are incident if x and y are incident in Γ and the partitions P_x, P_y are compatible.

Then by similar arguments as in the proof of Lemma 3 we get

COROLLARY 2. *There exists a surjective incidence and type preserving map*

$$\varphi\colon \mathscr{T} \to \bar{\mathscr{T}}$$

induced by $\varphi(x, P_x) = x$ *for* $(x, P_x) \in \mathscr{T}^{(1)}$. *Furthermore,* φ *maps each connected component* Θ *of* $\mathscr{T}$ *isomorphically onto the connected component* $\bar{\Theta} = \varphi(\Theta)$ *of* $\bar{\mathscr{T}}$ *and if* $\bar{\Theta}$ *is any connected component of* $\bar{\mathscr{T}}$ *then its preimage* $\varphi^{-1}(\bar{\Theta})$ *consists of three connected components of* $\mathscr{T}$ *which form an equivalence class of the above defined equivalence relation* $\equiv$.

2.2. *From* $(c.P)$- *to* T-*Geometries*

In this subsection we show how, under certain conditions, starting from a $(c.P)$-geometry Γ we can construct a tilde geometry with the same automorphism group.

Let us first assume that the rank of Γ is 4 and $G = \mathrm{Aut}(\Gamma) \cong M_{24}$. We will identify Γ with the geometry from [4, Example 2]. So Γ has the following sets of objects (where Ω, $\mathscr{T}$, and $\mathscr{O}$ are as in Example 1.1 the points, trios and octads of the Steiner system):

— $\Gamma^{(1)} = \{\{p, q\} \mid p, q \in \Omega,\ p \neq q\}$

— $\Gamma^{(2)} = \{\{a, b\} \mid a, b \in \Gamma_1,\ a \cap b = \varnothing\}$

— $\Gamma^{(3)} = \{(O_1, P, \{O_2, O_3\}) \mid O_i \in \mathcal{O}, \{O_1, O_2, O_3\} \in \mathcal{T}$, P is a partition of O_1 into four parallel lines in the affine geometry on $\Omega \backslash O_2\}$

— $\Gamma^{(4)} = \{(O, P) \mid O \in \mathcal{O}$, P is a partition of $\Omega \backslash O$ into parallel lines $\}$.

(Recall that for $O \in \mathcal{O}$ the complement $\Omega \backslash O$ bears the structure of an affine space over $GF(2)$ and $G_O \cong 2^4 A_8$ acts as affine group $AGL(4, 2)$ on $\Omega \backslash O$.)

Let Σ be a connected component of the structure $\mathcal{S}$ obtained from Γ by the shrinking construction described in the previous section. Then Σ has the diagram $\overset{1}{\circ}\overset{c}{\text{———}}\overset{2}{\circ}\overset{P^*}{\text{———}}\overset{3}{\circ}$. We have $G_2 \leqslant G_\Sigma$ and $G_2 \cong 2^{4+1}(\mathbf{Z}_2 \times \Sigma_5)$, and Lemma 2 and Corollary 1 yield that Σ is disconnected and that $O_2(G_2)$ is the kernel of the action of G_Σ on Σ. In particular, $O_2(G_2) \trianglelefteq G_\Sigma$, and the list of maximal subgroups of M_{24} in [3] shows that $G_\Sigma \leqslant G_S$, where $G_S \cong 2^6 3 \Sigma_6$ and G_S is the stabilizer of a sextet S in the Steiner system. From the structure of G_S we see that either $G_\Sigma = G_S$ or $G_\Sigma \cong 2^6(\mathbf{Z}_3 \times \Sigma_5)$. But in the latter $|G_\Sigma : G_2| = 3$, so Σ would just contain 3 points, which is not true (consider the residue of an element of type 3 in Σ). Hence $G_\Sigma = G_S \cong 2^6 3 \Sigma_6$.

Let $T^{(1)}$ be the set of connected components of $\mathcal{S}$, $T^{(2)}$ the set of triples of elements from $\Gamma^{(3)}$ of the shape

$$\{(O_1, P_1, \{O_2, O_3\}), (O_2, P_2, \{O_1, O_3\}), (O_3, P_3, \{O_1, O_2\})\}$$

such that $P_i \cup P_j$ is a partition into parallel lines of the affine space $\Omega \backslash O_k$ for all choices of $\{i, j, k\} = \{1, 2, 3\}$, and let $T^{(3)} = \Gamma^{(4)}$. Set $T = T^{(1)} \cup T^{(2)} \cup T^{(3)}$ and define an incidence relation on T in the way that $\Sigma \in T^{(1)}$ and $x \in T^{(2)} \cup T^{(3)}$ are incident if (x, P_x) (resp. (y, P_y) for all $y \in x$) belong to Σ, where P_x, P_y are suitable partitions of $\mathrm{res}(x)^-$ resp. $\mathrm{res}(y)^-$ into parallel lines of Γ. The elements in $T^{(3)}$ incident to a typical element

$$\{(O_1, P_1, \{O_2, O_3\}), (O_2, P_2, \{O_1, O_3\}), (O_3, P_3, \{O_1, O_2\})\} \in T^{(2)}$$

are the three elements $(O_i, P_j \cup P_k)$, $\{i, j, k\} = \{1, 2, 3\}$.

If $x = (O, P) \in T^{(3)}$ then $G_x \leqslant G_O \cong 2^4 A_8$. Since G_x stabilizes the partition P we see $G_x \cong 2^{1+6} L_3(2)$. If $x \in T^{(2)}$, x as above, then $G_x \leqslant G_t$ where G_t is the stabilizer of the trio $t = \{O_1, O_2, O_3\}$ and $G_t \cong [2^6](\Sigma_3 \times L_3(2))$. The Σ_3-factor permutes the three octads in t, so it will stabilize x as a set. The $L_3(2)$-factor can be viewed as the stabilizer of a partition of $\Omega \backslash O_i$ into two parallel planes for each of the O_i. Since G_x also stabilizes the partitions $P_j \cup P_k$ we deduce $G_x \cong [2^8](\Sigma_3 \times \Sigma_3)$. So we have shown that G_x is of the right shape for all $x \in T$. We leave it to the reader to convince himself that the incidence relation is defined in the right way to get a model of the tilde geometry for M_{24}.

Now let Γ be of arbitrary rank $n \geqslant 5$. For $1 \leqslant i \leqslant n-3$, let $\mathcal{S}_i$ be the incidence structure with sets of objects

— $\mathcal{S}_i^{(1)} = \Gamma^{(i+1)}$,

— $\mathcal{S}_i^{(j)} = \{(x, P_x) \mid x \in \Gamma^{(i+j)},\ P_x$ is a partition of $\mathrm{res}(x)^-$ into parallel objects of type $i\}$,

for $2 \leqslant j \leqslant n-i$, and an incidence relation defined similarly as before. Then $\mathcal{S}_2$ is just the structure $\overline{\overline{\mathcal{T}}}$ from the previous subsection and we see by induction that, for $i > 2$, each $\mathcal{S}_i$ is obtained in a similar way from $\mathcal{S}_{i-2}$. In other words and using Lemma 3, the connected components of $\mathcal{S}_i$ are isomorphic to the connected components of the incidence structure obtained by applying the shrinking construction to $\mathcal{S}_{i-1}$.

Assume now the connected components of $\mathcal{S}_{n-4}$ are the $(c.P)$-geometries for M_{24}. Let

— $T^{(i)} =$ connected components of $\mathcal{S}_i$, for $1 \leqslant i \leqslant n-3$,

— $T^{(n-2)} =$ triples of elements of $\Gamma^{(n-1)}$ which, for suitable partitions, are contained in a common connected component of $\mathcal{S}_{n-2}$ and there form a triple which corresponds to an element of type 2 of the T-geometry for M_{24},

— $T^{(n-1)} = \Gamma^{(n)}$

and let T be the geometry on $T^{(1)} \cup \cdots \cup T^{(n-1)}$ with the following incidence relation.

Two connected components $\Sigma_1 \in T^{(i)}$, $\Sigma_2 \in T^{(j)}$, $1 \leqslant i < j < n-2$, are incident if there are $x_i \in \Sigma_i^{(1)}$, $i = 1, 2$, such that x_1 and x_2 are incident in Γ. A connected component $\Sigma \in T^{(i)}$, $i < n-2$, and $y \in T^{(n-1)}$ are incident if there exists $x \in \Sigma^{(1)}$ such that x and y are incident in Γ and Σ is incident to $y \in T^{(n-2)}$ if the same holds for each of the three elements of $\Gamma^{(n-1)}$ in the triple y (of course with different x). Incidence between elements in $T^{(n-2)}$ and $T^{(n-1)}$ is defined by inclusion.

By construction the diagram of T is a string and by the assumption on $\mathcal{S}_{n-4}$ the geometry induced on the elements of types $n-2$ and $n-1$ in the residue of an element of type $n-3$ is the rank 2 tilde geometry for $3\Sigma_6$. On the other hand, by Corollary 2 there are exactly three elements of $T^{(1)}$ incident to an element of $T^{(2)}$ and these can be identified with the three different partitions into two parallel lines of the affine plane over $GF(2)$, i.e., with the three points on the projective line over $GF(2)$. By exactly the same arguments and induction, we see that the elements of type i in the residue of an element of type $j > i$ bijectively correspond to the i-dimensional subspaces in a j-dimensional $GF(2)$-space and that this correspondence provides in fact an isomorphism with a projective geometry over $GF(2)$. In particular, the residue of an element of type $n-1$ is isomorphic to the

$n-1$-dimensional projective $GF(2)$-space. So T is a tilde geometry of rank $n-1$.

Since the stabilizer of a suitable connected component $\Sigma \in T^{(1)}$ contains G_2, since $T^{(n-1)} = \Gamma^{(n)}$ and since $G = \langle G_2, G_n \rangle$, it is straightforward to see that G is a flag-transitive automorphism group of it. We formulate this result in the following

PROPOSITION 1. *Suppose G acts flag-transitively on a $(c.P)$-geometry Γ of rank $n \geqslant 5$. Apply the shrinking construction $n-3$ times on Γ and call the obtained incidence structures $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_{n-3}$. Suppose that $\mathcal{S}_1$ is disconnected, that the connected components of $\mathcal{S}_{i+1}$ are not equal to those of $\mathcal{S}_i$, $i = 1, ..., n-4$, and that the connected components of $\mathcal{S}_{n-4}$ are the $(c.P)$-geometries for M_{24}. Then G acts flag-transitively on a tilde geometry of rank $n-1$.*

Remarks. (1) The action of G on T is faithful: If $g \in G$ acts trivially on T then g fixes, in particular, every element in $T^{(n-1)} = \Gamma^{(n)}$. It is easy to see from Γ that then $g = 1$.

(2) One can also start from the rank 3 case and first construct the T-geometry for $3\Sigma_6$ from the $(c.P)$-geometry. This can be done, e.g., considering a model of the latter which exists inside $AGL(3,4)$ or getting a model via the shrinking process from the $(c.P)$-geometry for M_{24}. We did not choose this approach here because there was no obvious need for it, and it would have made the whole construction even more technical.

Notice that it is essential for the construction of a tilde geometry that the connected components of $\mathcal{S}_{n-3}$ are the $(c.P)$-geometries for $3\Sigma_6$. Otherwise their stabilizers might not act on a tilde geometry at all and one would not be able to construct even a rank-2-tilde geometry.

3. SOME PRELIMINARY LEMMAS

The first two lemmas might be well known, but, as we know no reference for them, we opted to provide the proofs.

LEMMA 4. *Let $G = L_4(2)$ act on an 8-dimensional $GF(2)$-vectorspace V which contains a G-invariant subspace V_1 such that V_1 and V/V_1 are nontrivial 4-dimensional modules for G. Suppose there exists $x \in V \setminus V_1$ such that $|x^G| = 15$. Then $V \neq \langle x^G \rangle$.*

Proof. Since $|x^G| = 15$, $C_G(x) \cong 2^3 L_3(2)$ is a maximal parabolic subgroup of G and each nontrivial coset of V/V_1 contains exactly one conjugate of x. Since $L_4(2)$ acts doubly transitively on the 4-dimensional module and hence also on the set of conjugates of x, it suffices to show that for some conjugate y of x the element xy is also conjugate to x. Then $\langle x^G \rangle = \{x^g \mid g \in G\} \cup \{1\}$, the assertion.

There exists an element $g \in G$ of order 3 acting fixed point freely on both 4-dimensional modules, hence g acts fixed point freely on V. Since $xx^g x^{g^2}$ is centralized by g, we get $xx^g x^{g^2} = 1$ and $xx^g = x^{g^2}$ has the desired property. ∎

LEMMA 5. *Let $G = M_{24}$ and V be a 12-dimensional $GF(2)$ G-module. If V contains a G-submodule V_1 with $|V_1| = 2^{11}$ and V_1 isomorphic to the (irreducible) Golay code module then V splits over V_1.*

Proof. Suppose not and consider the orbits of G on $V \backslash V_1$. Since 2048 is not divisible by 23, each element of order 23 in G must have a fixed point on $V \backslash V_1$. Now the list of maximal subgroups of M_{24} in [3] shows that G has an orbit of length 24. Since any two subgroups of G isomorphic to M_{23} intersect in an M_{22} then the sum of any two vectors in this orbit is fixed by a subgroup of G isomorphic to M_{22}. But M_{22} does not stabilize a nontrivial vector in V_1, a contradiction. ∎

In the next three lemmas G is a flag-transitive automorphism group of some $(c.P)$- or $(c.T)$-geometry Γ satisfying (*); all other notation is as in the Introduction.

LEMMA 6. *Suppose there exists an involution $x \in G_2 \backslash G_1$ such that $[G_{12}, x] = 1$. Let $X_1 = \langle x^{G_1} \rangle$ and $X = \langle x^{G_{1n}} \rangle$. If X is elementary abelian and $2^{n-1} \leqslant |X| \leqslant 2^n$ then $|X| = 2^{n-1}$ and X_1 is isomorphic to a factor group of the universal representation group of* $\operatorname{res}(\alpha_1)$.

Proof. If $p \in \operatorname{res}(\alpha_1) \cap \Gamma^{(2)}$ then $p = \alpha_2^g$ for a suitable $g \in G_1$ and we set $u_p = x^g$. Since $[G_{12}, x] = 1$ this is well-defined and it suffices to show that $u_p u_q u_r = 1$ whenever p, q, r are incident to a common element $l \in \operatorname{res}(\alpha_1) \cap \Gamma^{(3)}$. (Then X_1 satisfies the axioms of Definition 1.)

By flag-transitivity we may assume $l = \alpha_3$, $p = \alpha_2$. Then $l \in \operatorname{res}(\alpha_n)$, q and r are conjugate to p in G_{13n} and $\langle u_p, u_q, u_r \rangle \leqslant X$. As $x \notin K_n$ and $G_n/K_n \cong 2^{n-1}L_{n-1}(2)$ we have $G_n = XG_{1n}$. So either $|X| = 2^{n-1}$ or $|X \cap K_n| = 2$, and since $X \cap K_n \trianglelefteq G_n$, in the latter case we get $X \cap K_n \leqslant Z(G_n)$. Furthermore, as $C_{G_{1n}}(x) = G_{1n} \cap G_2$, the action of G_{1n} on $X/X \cap K_n$ corresponds to the action on the projective $GF(2)$-space $\operatorname{res}(\{\alpha_1, \alpha_n\})$. Since α_3 corresponds to a line in this space the assertion holds in $X/X \cap K_n$, particularly it holds if $|X| = 2^{n-1}$. Suppose $1 \neq z = u_p u_q u_r \in X \cap K_n \leqslant Z(G_n)$. Let $X_3 = \langle x^{G_3} \rangle$. Since $G_3 = G_{123} G_{3\ldots n}$ and $[x, G_{123}] = 1$ we have $X_3 = \langle x^{G_{3\ldots n}} \rangle = \langle u_p, u_q, u_r \rangle$. Now X_3 is invariant under P_n, $z \in X_3$, and as $z \notin Z(G)$ we have $[z, P_n] \neq 1$. However, we see in G_1 that $z^{P_n} \subseteq B$. Since $X \cap B = X \cap K_n$ this yields a contradiction. ∎

LEMMA 7. *We have* $K_i = K_{1i}$ *for* $i \geqslant 2$.

Proof. Let $g \in K_{1i}$. Then g fixes all elements in $\mathrm{res}(\alpha_1) \cap \mathrm{res}(\alpha_i)$. Let $p \in \Gamma^{(1)}$ be incident to α_i. Then there exists $l \in \Gamma^{(2)}$ incident to α_i such that $\mathrm{res}(l) \cap \Gamma^{(1)} = \{\alpha_1, p\}$. Now g fixes α_1 and l, so g must fix p. This implies that g acts trivially on $\mathrm{res}(\alpha_i)$ and hence $g \in K_i$. The other inclusion is trivial. ∎

LEMMA 8. *If* $n \leqslant 5$ *then* $K_1 = 1$. *If* $n = 6$ *then* $|K_1| \leqslant 2$.

Proof. By [7] we know that, if Γ is of $(c.P)$-type, then $G_1/K_1 \cong M_{23}$ or Co_2, if $n = 5$, and $G_1/K_1 \cong BM$, if $n = 6$. If Γ is of $(c.T)$-type, then $G_1/K_1 \cong M_{24}$, Co_1, M for $n = 4, 5, 6$, respectively. We will make use of the descriptions of the corresponding residual geometries as given in [6].

Let us call the elements of Γ of type 1 "points" and the elements of type 2 "lines." Then each line consists of 2 points and the relation "having the same points" is an equivalence relation on the set of lines on the point α_1, which is preserved by G_1. Since in the considered cases G_{12}/K_1 is a maximal subgroup of G_1/K_1, the action of G_1/K_1 on that set is primitive. Hence there are no two lines with the same points. In particular, $G_p \cap G_q \leqslant G_l$ for two points p, q on a line l.

Now let $g \in K_1$, p a point collinear to α_1 and l the line on α_1 and p. Then g fixes l and α_1, hence $g \in G_p$. Let k be a line on p such that l and k are incident to a common element x of type 3 and q the other point on k. Then x is incident to α_1 and there is a line through α_1 and q in $res(x)$. So g fixes q and $g \in G_p \cap G_q \leqslant G_k$. This means that g fixes all lines incident to p which are coplanar with l. In terms of the residual geometry $res(p)$, where l corresponds to a point, this means that g fixes all points collinear to l. Since the points of the P-geometry for M_{23} are just the 23 points of the Steiner system $\mathscr{S}(23, 7, 4)$, the lines are all the pairs of different points, and incidence is defined by inclusion, any two points are collinear in this case. So K_1 fixes all points of the Steiner system and $K_1 \leqslant K_p$. Hence $K_1 = K_p = 1$. In the M_{24}-geometry l corresponds to a sextet of the Steiner system and g must fix all sextets which refine a common trio with l. So we get $K_1 \leqslant K_p$ again. For the geometries of the Conway groups one can deduce the same from the descriptions in terms of the Leech lattice given in Example 2 resp. [6].

Finally, also by [6] the points of the geometries for BM and M can be identified with the conjugacy class of 2-central involutions and the involutions corresponding to the set of points collinear to l generate the subgroup $O_2(G_{pl}/K_p)$. Since this is an extraspecial group of order 2^{1+22} resp. 2^{1+24} we get $|K_1 : K_1 \cap K_p| \leqslant 2$ in those cases. So it remains to show that $K_1 \cap K_p = 1$. From Lemma 7 and $G_6/K_6 \cong 2^5 L_5(2)$ we see that $K_1 K_6/K_6$ and $K_1 K_6/K_1$ are 2-groups. Hence K_1 and K_6 are 2-groups by the Frattini argument. Since B/K_1 is also a 2-group, we get that B and all K_i are 2-groups. Suppose first that $C_{G_1}(K_1) \leqslant K_1$. Let $S \in Syl_2(G_1 \cap G_2)$ and $Z_1 = \langle \Omega_1(Z(S))^{G_1} \rangle$. Notice that $|G_2 : G_1 \cap G_2| = 2$, so $G_1 \cap G_2 \trianglelefteq G_2$ and $G_2 = (G_1 \cap G_2)\, N_{G_2}(S)$. If $Z_1 = \Omega_1(Z(S))$ then $Z_1 \trianglelefteq \langle G_1, N_{G_2}(S) \rangle = G$, which is impossible. So $Z_1 \neq \Omega_1(Z(S))$ and Z_1 involves a non-trivial G_1/K_1-module. By [1, (11.1)] BM and M do not have F-modules. Hence $J(S) \leqslant K_1$ (where $J(S)$ is the Thompson subgroup of S). But then $J(S) = J(K_1) \trianglelefteq \langle G_1, N_{G_2}(S) \rangle = G$, again a contradiction. So we have $C_{G_1}(K_1) \nleqslant K_1$, hence $G_1 = C_{G_1}(K_1)\, K_1$ and, by symmetry, $G_p = C_{G_p}(K_p)\, K_p$. Therefore $K_1 \cap K_p$ is normal in $\langle O^2(G_1), O^2(G_p), K_1, K_p \rangle = \langle G_1, G_p \rangle$. But as $G_6 = \langle G_1 \cap G_6, G_p \cap G_6 \rangle$ we have $G = \langle G_1, G_6 \rangle \leqslant \langle G_1, G_p \rangle$. So $K_1 \cap K_p \trianglelefteq G$ and $K_1 \cap K_p = 1$. This proves the assertion. ∎

4. CLASSIFICATION OF THE GEOMETRIES

LEMMA 9. *If Γ is a $(c.P)$-geometry of rank 5 and $G_1 \cong M_{23}$, then $G \cong M_{24}$.*

Proof. It is sufficient to show that $|G : G_1| = |\Gamma^{(1)}| = 24$. Since $G_{12} \cong M_{22}$, we have $|\mathrm{res}(\alpha_1)^{(2)}| = |G_1 : G_{12}| = 23$. So there are 23 points which are collinear to α_1 and it suffices to show that, if p and q are collinear to α_1, then p is collinear to q. Let $l_p, l_q \in \Gamma^{(2)}$ be the lines through α_1 and p resp. q. Since l_p, l_q correspond to points in the geometry for M_{23} and any two points in this geometry are collinear by the 2-transitive action of M_{23}, we see in $\mathrm{res}(\alpha_1)$ that there exists $x \in \Gamma^{(3)}$ such that x is incident to l_p and l_q. But now p and q are incident to x, and since the elements of $\mathrm{res}(x)^-$ form an affine space, the claim follows. ∎

For the rest of the section we fix the following situation: From Lemma 7 we know that $K_2 = K_{12}$. Further we have $|G_2 : G_{12}| = 2$. So $G_{12} \trianglelefteq G_2$ and there exists $x \in G_2 \backslash G_{12}$ such that $x^2 \in G_{12}$. Since $K_2 = O_2(G_{12})$ we see that x acts on G_{12}/K_2. But except in one case, where we will have $G_{12}/K_2 \cong 3\Sigma_6$, the group G_{12}/K_2 will not possess outer automorphisms of order 2, so $G_2/K_2 \cong \mathbf{Z}_2 \times G_{12}/K_2$ and we may assume $[x, G_{12}] \leqslant K_2$. If $G_{12}/K_2 \cong 3\Sigma_6$ then $|\mathrm{Out}(G_{12}/K_2)| = 2$ and the outer automorphism of $3\Sigma_6$ interchanges the two conjugacy classes of subgroups of type $\mathbf{Z}_2 \times \Sigma_4$. But these correspond to stabilizers of elements of different types in $\mathrm{res}(\alpha_2)^+$ in Γ, and as $x \in \mathrm{Aut}(\Gamma)$, x preserves the types. So x cannot act as outer automorphism and we get $[x, G_{12}] \leqslant K_2$ in this case as well.

Unless stated otherwise, from now on x will always denote such an element. Then $[x, B] \leqslant [x, G_{12}] \leqslant K_2 \leqslant B$ and $x \in N_{G_2}(B)$. If Γ is of $(c.T)$-type this yields $x \in P_1$; if Γ is of $(c.P)$-type then $x \in P_1 P_n$. Furthermore, by the Three Subgroup Lemma G_{12} acts on $[K_2, x]\, K_2'$. In particular, if K_2 is elementary abelian and irreducible as G_{12}/K_2-module, then $[K_2, x] = 1$. Further, since then $[G_{12}, x^2] = 1$ we get that x is of order 2.

LEMMA 10. *If Γ is a (c.T)-geometry of rank 4 and $G_1 \cong M_{24}$, then $G \cong 2^{11}{:}\, M_{24}$ and $O_2(G)$ is the universal representation group of the T-geometry for M_{24} (so it is the Golay code module).*

Proof. We have $G_{12} \cong 2^6 3 \Sigma_6$ where K_2 is an irreducible G_{12}/K_2-module. So as just remarked, $Q_2 = \langle K_2, x \rangle$ must be an elementary abelian group of order 2^7. Let $\rho \in O_{23}(G_{12})$, $o(\rho) = 3$. Then ρ acts fixed point freely on K_2. So $C_{Q_2}(\rho)$ must be of order 2. Since $C_{Q_2}(\rho)$ is G_{12}-invariant we may assume $[x, G_{12}] = 1$.

Now consider G_4. We have $G_{14} \cong 2^{1+6} L_3(2)$, $G_4/K_4 \cong 2^3 L_3(2)$, and $G_4 = G_{14} X$ where $X = \langle x^{G_{14}} \rangle$. Since $K_4 \leqslant G_{12}$ we get $[X, K_4] = \langle [x, K_4]^{G_{14}} \rangle = 1$. As K_4 is extraspecial this yields $|X \cap K_4| \leqslant 2$ and $|X| \leqslant 16$. Since X is generated by involutions and $G_4/O_2(G_4) \cong L_3(2)$ acts transitively on $(X/X \cap K_4)^{\#}$, we see that X must be elementary abelian. So we can apply Lemma 6 to see that $X_1 = \langle x^{G_1} \rangle$ is a representation group for the tilde geometry for M_{24}. Hence $|X_1| = 2^{11}$ by [6, (3.2)] resp. [8]. Since X_1 is normalized by G_1 and $G = \langle x, G_1 \rangle$, we get $G = X_1 G_1$ and the lemma is proved. ∎

In the rank 5 and rank 6 cases we will consider (c.P)- and (c.T)-geometries at the same time. This has the following reason: Many of our calculations mainly depend on the 2-structures of some of the maximal parabolic subgroups. Since the P-geometries for Co_2 and BM can be defined as certain subgeometries in the T-geometries for Co_1 resp. M, these structures are quite similar for both classes of geometries. In particular, it turned out that often we just can use the same arguments without distinguishing between T- and P-geometries.

LEMMA 11. *Let* rank$(\Gamma) = 5$.

(a) *If Γ is of (c.P)-type and $G_1 \cong Co_2$, then either $G \cong 2^{22}{:}\, Co_2$ resp. $2^{23}{:}\, Co_2$ and $O_2(G)$ is a representation group of the P-geometry for Co_2 (so it is isomorphic to a section of the Leech lattice* mod 2); *or $G \cong Co_1$.*

(b) *If Γ is of (c.T)-type and $G_1 \cong Co_1$, then $G \cong 2^{24}{:}\, Co_1$ and $O_2(G)$ is the universal representation group of the tilde geometry for Co_1 (so it is isomorphic to the Leech lattice* mod 2).

Proof. From the geometries of Co_2 and Co_1 we know that $G_{12} \cong 2^{10}\mathrm{Aut}(M_{22})$ resp. $2^{11} M_{24}$ and that K_2 is the irreducible Golay code module for G_{12}/K_2. So $[x, K_2]=1$ and $O_2(G_2)=\langle x, K_2\rangle$ is an elementary abelian group of order 2^{11} resp. 2^{12} by the above remark. Now if $G_{12}/K_{12}\cong \mathrm{Aut}(M_{22})$, then we have two general possibilities for the structure of G_2:

(a.1) $G_2\cong \mathbf{Z}_2\times 2^{10}\mathrm{Aut}(M_{22})$,

(a.2) $G_2\cong 2^{10+1}\mathrm{Aut}(M_{22})$, where the action of $G_2/O_2(G_2)$ on $O_2(G_2)$ is non-split,

whereas in (b) by Lemma 5 we must have $G_2\cong \mathbf{Z}_2\times 2^{11}M_{24}$.

Next look at G_5. We have $G_5/K_5\cong 2^4L_4(2)$. From the structure of G_1 we know that in (a) $G_{15}\cong(2^{1+6}\times 2^4)\,L_4(2)$, $|Z(K_5)|=2^5$ and $|K_5\cap K_2|=2^7$, and in (b) $G_{15}\cong 2^{1+8+6}L_4(2)$, $|Z(K_5)|=2$, and $|K_5\cap K_2|=2^8$.

Let us first consider (a.1) and (b). In these cases $[x, K_5]=1$ and $x\in N_{G_2}(K_5)=G_2\cap G_5$. Hence if we set $X=\langle x^{G_{15}}\rangle$, then $G_5=XG_{15}$, $[X, K_5]=1$, and $X\cap K_5\leqslant Z(K_5)$. So $|X|\leqslant 2^9$ resp. 2^5. As $Z(K_5)$ is elementary abelian and $o(x)=2$ every element in the coset $xZ(K_5)$ must be an involution. Of course, the same holds for any conjugate of x under G_{15}. So the irreducible action of $G_{15}/K_5\cong L_4(2)$ on $X/X\cap K_5$ shows that each element in $X\backslash X\cap K_5$ must be an involution and hence X must be elementary abelian. In (a.1) now we apply Lemma 4 to see that X splits over the 4-dimensional module in $Z(K_5)$. So we can choose x in such a complement and then in both cases $|X|\leqslant 32$. Now Lemma 6 yields that $X_1=\langle x^g\mid g\in G_1\rangle$ is a representation group for the residual geometry of G_1/K_1. So X_1 is elementary abelian of order 2^{22} or 2^{23} in (a.1) and of order 2^{24} in (b) by [6, (5.7), (5.8)]. Again because X_1 is normalized by G_1 and $G=\langle x, G_1\rangle$ we get that $G=X_1G_1$ is as stated.

Now consider (a.2). Suppose $[y, K_5]\leqslant Z(G_{15})$ for all $y\in O_2(G_2)\backslash K_2$. Because $O_2(G_2)=\langle y\mid y\in O_2(G_2)\backslash K_2\rangle$, in that case we would get that $[O_2(G_2), K_5]\leqslant Z(G_{15})$. But K_5K_2/K_2 does not induce transvections on K_2. So we get a contradiction. Hence we may choose x with $[x, K_5]\nleqslant Z(G_{15})$. Then, in particular, $[X, K_5]=[\langle x^{G_5}\rangle, K_5]=\langle [x, K_5]^{G_5}\rangle$ is not contained in $K_5\cap O_2(G_2)$.

Now let Σ be a connected component of the structure $\mathscr{S}$ obtained from Γ by shrinking as described in Section 2.. We can choose Σ such that the flag $\{\alpha_2, ..., \alpha_n\}$ maps onto a maximal flag in Σ. Let $H=G_\Sigma$ and let N be the kernel of the action of H on Σ. By Lemma 2 we have $N=O_2(G_2)$ and, by construction of $\mathscr{S}$, $O_2(G_5)\,N/N=O_2(H_4/N)$. Since $[X, K_5]\nleqslant N$, we see that $O_2(H_4/N)$ is non-abelian. So [4] implies that $H/N\cong M_{24}$. Now G acts flag-transitively on a rank 4 tilde geometry by Proposition 1, and since H is the stabilizer of an object of this geometry, by [7] we get $G\cong Co_1$. ∎

LEMMA 12. *Let* $\operatorname{rank}(\Gamma)=6$.

(a) *If* Γ *is of* $(c.P)$-*type and* $G_1/K_1 \cong BM$, *then either* G *is isomorphic to a section of* $(2\cdot BM * 2\cdot BM)$: $\mathbf{Z}_2$ *or* $G\cong M$.

(b) *If* Γ *is of* $(c.T)$-*type and* $G_1/K_1 \cong M$, *then* $G\cong M\times M$ *or* $M \wr \mathbf{Z}_2$.

Proof. By Lemma 8 we have $|K_1|\leqslant 2$. Hence we get the following possibilities for G_1:

(a.1) $K_1=1$, $G_1 \cong BM$,

(a.2) $|K_1|=2$, $G_1\cong \mathbf{Z}_2\times BM$,

(a.3) $|K_1|=2$, $G_1\cong 2\cdot BM$, a non-split extension,

respectively

(b.1) $K_1=1$, $G_1\cong M$,

(b.2) $|K_1|=2$, $G_1\cong \mathbf{Z}_2\times M$ (as the Schur multiplier of M is trivial by [3]).

We set $\langle z\rangle = Z(G_1)=K_1$. Then $G_{12}/\langle z\rangle \cong 2^{1+22}Co_2$ resp. $2^{1+24}Co_1$.

Let us first assume we are not in case (a.3). Then $Q_2=[K_2, G_{12}]$ is extraspecial and $[K_2, Q_2]=Q_2'$. Since $[x, G_{12}, Q_2]\leqslant [K_2, Q_2]=Q_2'$ and $[Q_2, x, G_{12}]\leqslant [K_2, G_{12}]=Q_2$, the Three Subgroup Lemma yields $[Q_2, x]=[G_{12}, Q_2, x]\leqslant Q_2$. Hence $Q_2 \trianglelefteq G_2$. Now also $C_{G_2}(Q_2)\trianglelefteq G_2$ and $[C_{G_2}(Q_2), O^2(G_{12})]=1$. Further, as $[x, G_{12}, Q_2]\leqslant Q_2'$ another application of the Three Subgroup Lemma shows that

$$[x, Q_2, G_{12}]\leqslant [Q_2, G_{12}, x]\, Q_2' = [Q_2, x]\, Q_2',$$

i.e., G_{12} acts on $[Q_2, x]\, Q_2'$. Since $[Q_2, x]\, Q_2' < Q_2$ and Q_2/Q_2' is an irreducible G_{12}/Q_2-module we must have $[x, Q_2]\leqslant Q_2'$; and since Q_2 is extraspecial we can choose x such that $[x, Q_2]=1$. In particular, $|C_{G_2}(Q_2)|=4$ resp. 8 (depending whether we have $K_1=1$ or not).

Now $o(x)\leqslant 4$ (since $C_{G_{12}}(Q_2)=K_1Q_2'$ is elementary abelian) and x centralizes a Sylow-2-subgroup $S\leqslant G_1'$. In particular x centralizes $O_2(H_{16})$, where $H_{16}=G_6\cap G_1'$, which implies $x\in N_{G_2}(O_2(H_{16}))=G_2\cap G_6$. Notice that $H_{16}/O_2(H_{16})\cong L_5(2)$ with $|O_2(H_{16})|=2^{30}$ resp. 2^{36}, and that $G_6=\langle z\rangle H_{16}X$ where $X=\langle x^{H_{16}}\rangle$. Now XK_6/K_6 is an elementary abelian group of order 2^5. Furthermore

$$[X, O_2(H_{16})]=[\langle x^{H_{16}}\rangle, O_2(H_{16})]=\langle [x, O_2(H_{16})]^{H_{16}}\rangle = 1$$

and hence $X\cap O_2(H_{16})\, X\leqslant Z$, where $Z=Z(O_2(H_{16}))$. As $|Z|=2^5=|X/X\cap O_2(H_{16})|$ we have $|ZX|=2^{10}$ and $H_{16}/O_2(H_{16})\cong L_5(2)$ acts irreducibly on both z and XZ/Z. Now if $o(x)=4$ then $X=ZX\cong \mathbf{Z}_4^5$. But by [5] $GL_5(\mathbf{Z}_4)$ does not contain a subgroup isomorphic to $L_5(2)$ and we know from G_1 that H_{16} splits over $O_2(H_{16})$, which yields a contradiction. So $o(x)=2$ and the action of $L_5(2)$ implies that X is elementary abelian.

We claim that XZ splits over Z. We know that Z and $XK_6/K_6 \cong X/(X\cap K_6)=X/(X\cap Z)\cong XZ/Z$ are isomorphic as $L_5(2)$-modules. Consider a subgroup L of $L_5(2)$ of order $31\cdot 5$. Since L acts completely reducibly on XZ, we see that L has three orbits of length 31 and 6 orbits of length 155 on $(XZ)^{\#}$ where each of the three orbits of length 31 generates a 5-dimensional subspace of XZ. As x has exactly 31 conjugates under H_6 the claim follows, i.e., X is elementary abelian of order 2^5. Replacing G_1 by G_1' in Lemma 6 now we get that $X_1=\langle X^{G_1'}\rangle$ is a representation group for the residual geometry for $G_1'\cong G_1/K_1$. Hence $X_1\cong BM$ or $2\cdot BM$ resp. M by [6].

Consider the semidirect product X_1G_1'. If $z=1$ then $X_1G_1'=X_1G_1=G$ and since $\mathrm{Out}(G_1)=1$ we get $G\cong G_1\times G_1$. If $z\neq 1$ then $[z,x]\neq 1$ (otherwise we would get $z\in Z(\langle x, G_1\rangle)=Z(G)$. But as $z\in K_6$ we have $[z,X]\leqslant K_6$ and $[z,X_1]\leqslant X_1G_1'$. So z normalizes $X_1G_1'\cong X_1 * X_1$ but not X_1 and we get the first case of (a) resp. (b). (Notice that G_1 is the diagonal subgroup of G.)

Now consider case (a.3). Let Σ be a connected component of the structure $\mathscr{S}$ as described in Section 2. Let $H=G_\Sigma$ and N the kernel of the action of H on Σ. By Lemma 2 we have $N=K_2\langle x\rangle$, and by Lemma 11 we have $H/N\cong Co_1$ or $2^{22}Co_2$ (resp. $2^{23}Co_2$).

If $H/N\cong Co_1$, then Proposition 1 yields that G acts flag-transitively on a rank 5 tilde geometry. So $G\cong M$ by [7] in this case.

Suppose $H/N\cong 2^{22}Co_2$ (resp. $2^{23}Co_2$). Since G_{12} acts on $[K_2,x]\,K_2'$, the structure of K_2 as G_{12}/K_2-module implies that $[x,K_2]\leqslant Z(G_{12})$. Hence $|K_2\colon C_{K_2}(x)|\leqslant 4$. If $[K_2,x]\nleqslant K_2'$, then $|K_2/K_2' : C_{K_2/K_2'}(x)|=|[K_2/K_2', x]|=2$, but K_2/K_2' does not contain a G_{12}-submodule of index 2. So $[K_2,x]\leqslant K_2'$, and since $N=\langle x,K_2\rangle$, $N'=K_2'$ is of order 2. As x^2 centralizes K_2 we get $o(x)\leqslant 4$ again. As above $[x,z]\neq 1$, so $Z(N)=N'$ and N is extraspecial of order 2^{1+24}.

Let $N_1=O_2(H)$. Then N_1 acts on N/N' and $C_{N/N'}(N_1)\neq 1$. Let C be the full preimage in N of $C_{N/N'}(N_1)$. Then $C\trianglelefteq H$ and the action of Co_2 on N/N' implies $z\in C$. Therefore, $[z,N_1]\leqslant N'$ and $|N_1\colon C_{N_1}(z)|\leqslant 2$. Since N_1/N does not contain a G_{12}-submodule of index 2, we get $N_1=C_{N_1}(z)\,N$. Now consider the action of N_1/N on Σ. Because N acts trivially on Σ and N_1/N acts regularly on the points of Σ, i.e., on $\Sigma\cap\Gamma^{(2)}$, we see that even $C_{N_1}(z)$ acts regularly. We may assume $\alpha_2\in\Sigma$. Let $\beta_2\in\Gamma^{(2)}$ be the line parallel to α_2 in $\mathrm{res}(\alpha_3)$. Our construction implies that $\beta_2\in\Sigma$. Choose $n\in C_{N_1}(z)$ such that $\beta_2=\alpha_2^n$ and let $\beta_1=\alpha_1^n$. (Notice that $\beta_1\neq\alpha_1$ because β_1 is incident to β_2, and α_2 and β_2 are parallel.) Then

$$K_{\beta_1}=K_{\alpha_1^n}=K_1^n=\langle z\rangle^n=\langle z\rangle=K_1.$$

But $\alpha_1, \beta_1 \in \mathrm{res}(\alpha_3)$, so α_1 and β_1 lie on a common line $l \in \Gamma^{(2)} \cap \mathrm{res}(\alpha_3)$. By flag-transitivity α_1 and β_1 are conjugate in G_l, so $G_l \leqslant N_G(K_1)$ and $K_1 \trianglelefteq \langle G_1, G_l \rangle = G$, a contradiction. ∎

Proof of Theorems 1 *and* 2. The three cases (a), (b), and (c) of Theorem 1 follow from the Lemmas 9, 11(a), and 12(a), respectively, while those of Theorem 2 follow from Lemmas 10, 11(b), and 12(b). ∎

Remark. Our proofs also show that the amalgam of maximal parabolics is always uniquely determined. On the other hand, it follows from the structures of the obtained groups that there will be just one way (up to conjugation) to embed the amalgam into the group. This means that the geometries are uniquely determined up to isomorphism, too.

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